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CALCULATING AN ACOUSTIC FIELD IN LAMINAR-HETEROGENEOUS MEDIA

V. M. Kudryashov

The practice of calculating an acoustic field shows that the better the components (whose sum describes the field to be calculated) correspond to the spatial structure of the field, the more effective the selected means of calculation.

Let us examine a field-calculation method convenient in the region where the zonality in the field structure applicable to uniaxial channels appeared. For this let us introduce a cylindrical coordinate system (r, Θ, z) (Fig. 1) and examine the acoustic field in a layer $h_0 \leq z \leq H$ filled with a medium which has no shear elasticity, in which the square of the refraction index $n^2(z)$ and its derivatives of an order no lower than the first are continuous functions of the z coordinate. In the half-space $z \geq H$ we set $n^2(z) = n^2(H) + 2a(z - H)$, $n(H - 0) \equiv n(H + 0)$. The acoustic potential (of velocities) $\Psi(r, z, z_0) \exp(-i2\pi f_0 t)$ should satisfy the wave equation everywhere except point $r = 0$, $z = z_0$ in which there is located the point source of a sound operating under stationary conditions at frequency $f_0 \gg C(z)|(d/dz)h^2(z)|$; in the vicinity of the source

$$\Psi(r, z, z_0) \quad \exp(ik_0L)/L,$$

where

$$L = \sqrt{r^2 + (z - z_0)^2} \to 0.$$

In addition, $\lim\limits_{L\to\infty} \Psi(r, z, z_0) \to 0$. The medium is bounded from above by the plane $z = h_0$ on which $\Psi(r, h_0, z_0) = 0$.

In this case, according to L. M. Brekhovskikh [1],

$$\begin{gathered}\Psi(r, z, z_0) = \frac{i\pi k_0^2}{4} \int\limits_{-\infty+i\epsilon}^{\infty+i\epsilon} [\Phi_1(z_0;\ \varkappa) - p(\varkappa)\Phi_2(z_0;\ \varkappa)] \times \\ \times [\Phi_2(z;\ \varkappa) - q(\varkappa)\Phi_1(z;\ \varkappa)]\,[1 - p(\varkappa)q(\varkappa)]^{-1} H_0^{(1)}(k_0 r\varkappa)\,\varkappa\, d\varkappa,\end{gathered} \tag{1}$$

where $\Phi_1(z; \kappa)$ and $\Phi_2(z; \kappa)$ are two linearly independent solutions of the equation

$$\frac{d^2}{dz^2}\Phi(z) + k_0^2\,[n^2(z) - \varkappa^2]\,\Phi(z) = 0. \tag{2}$$

Magnitudes $q(\kappa)$ and $p(\kappa)$ are determined from the boundary conditions on the upper and lower boundaries of the medium, respectively, or from radiation conditions if the medium is unbounded. In the case in question we get

$$\begin{gathered}q(\varkappa) = \Phi_2(h_0;\ \varkappa)/\Phi_1(h_0;\ \varkappa);\\ p(\varkappa) = \\ = \frac{\Phi_1(H;\ \varkappa) - \frac{\rho_g}{\rho}\left[H^{(1)}_{1/3}(\eta_H)/H^{(1)}_{2/3}(\eta_H)\right] k_0^{-1}\,[n^2(H) - \varkappa^2]^{-1/2}\left[\frac{\partial}{\partial z}\Phi_1(z;\varkappa)\right]_{z=H}}{\Phi_2(H;\ \varkappa) - \frac{\rho_g}{\rho}\left[H^{(1)}_{1/3}(\eta_H)/H^{(1)}_{2/3}(\eta_H)\right] k_0^{-1}\,[n^2(H) - \varkappa^2]^{-1/2}\left[\frac{\partial}{\partial z}\Phi_2(z;\ \varkappa)\right]_{z=H}};\\ \eta_H = \frac{k_0}{3b}\,[n^2(H) - \varkappa^2]^{3/2}.\end{gathered}$$

The integration contour in equation (1) circumvents those positions from below in which Re $\kappa \geq 0$, Im $\kappa \geq 0$, and circumvents the remaining singular points of a subintegral function from above.

If $z < h$, $(d/dz)n^2(z) > 0$, then

$$\begin{gathered}\Phi_1(z;\ \varkappa) = f(\varkappa)\sqrt{\frac{2}{\pi k_0}}\,[n^2(z) - \varkappa^2]^{-1/4}\exp\left[i\omega - i\frac{5\pi}{12}\right)(1 + \Lambda_1),\\ -\frac{\pi}{2} \leqslant \arg\omega \leqslant \frac{3\pi}{2};\\ \Phi_1(z;\ \varkappa) = \sqrt{\frac{2}{\pi k_0}}\,[n^2(z) - \varkappa^2]^{-1/4}\Big[f(\varkappa)\exp\left(i\omega - i\frac{5\pi}{12}\right)(1 + \Lambda_1) - \\ - f^{-1}(\varkappa)\exp\left(i\omega + i\frac{\pi}{12}\right)(1 + \Lambda_2)\Big],\quad -\frac{3\pi}{2} \leqslant \arg\omega \leqslant -\frac{\pi}{2};\end{gathered} \tag{3}$$

$$\Phi_2(z;\varkappa)=f^{-1}(\varkappa)\sqrt{\frac{2}{\pi k_0}}\,[n^2(z)-\varkappa^2]^{-1/4}\exp\left(-i\omega+i\frac{5\pi}{12}\right)(1+\Lambda_2),$$
$$-\frac{3\pi}{2}\leqslant\arg\omega\leqslant\frac{\pi}{2};$$
$$\Phi_2(z;\varkappa)=\sqrt{\frac{2}{\pi k_0}}\,[n^2(z)-\varkappa^2]^{-1/4}\Big[f(\varkappa)^{-1}\exp\left(-i\omega+i\frac{5\pi}{12}\right)(1+\Lambda_2)-$$
$$-f(\varkappa)\exp\left(i\omega-i\frac{\pi}{12}\right)(1+\Lambda_1)\Big],\quad \frac{\pi}{2}\leqslant\arg\omega\leqslant\frac{3\pi}{2}. \tag{3}$$

In formulas (3) the designations

$$\omega=\omega(z;\varkappa)=k_0\int_{z_1}^{z}\sqrt{n^2(\xi)-\varkappa^2}\,d\xi,$$

are used, where z_1 is the root of the equation $n(z) = \kappa$, which satisfies the requirement $z_1 \leq h$, if

$$\operatorname{Im} z_1=0;\quad \lim_{\varkappa^2\to n^2(h)}z_1=\lim_{\varepsilon\to 0}(h-\varepsilon).$$

Since $\omega(z; \kappa)$ is a multivalued function of κ with branching points $\kappa = \pm h(z)$, we assume

$$\arg\omega=0,\quad \text{at}\quad \varkappa^2<n^2(z)<1,\quad \operatorname{Im}\varkappa^2=0;$$
$$\arg\omega=\frac{3\pi}{2},\quad \text{at}\quad n(z)<\varkappa<1,\quad \operatorname{Im}\varkappa=\lim_{\varepsilon\to 0}(0-\varepsilon);$$
$$\arg\omega=-\frac{3\pi}{2},\quad \text{at}\quad n(z)<\varkappa<1,\quad \operatorname{Im}\varkappa=\lim_{\varepsilon\to 0}(0+\varepsilon).$$

The whole picture is symmetrical relative to the point $\kappa = 0$. The introduction of factor $f(\kappa)$ into formulas (3) is associated with the presence of a maximum $n^2(z)$ at $z = h$; $f(\kappa)$ assures a regular behavior of the right sides of expressions (3) on the entire plane κ.

$$f(\varkappa)=\left[\Gamma\left(\frac{\omega_2}{\pi}+\frac{1}{2}\right)\right]^{1/2}(2\pi)^{-1/4}\exp\left\{-\frac{\omega_2}{2\pi}\ln\frac{\omega_2}{\pi e}\right\},$$

where $\omega_2 = \omega(z_2; \kappa)$; z_2 is the second root of the equation $n(z) = \kappa$ $z_2 \geq h$ at $\operatorname{Im} z_2 = 0$; $f(\kappa) \approx 1$ if $|\omega_2| \gg 1$, $|\arg \omega_2| < \pi$. And, finally,

$$\Lambda_i(\varkappa)=O\left(\frac{1}{\omega}\right);\quad |\Lambda_i(\varkappa)|\ll 1,\quad |\omega|>1.$$

Asymptotic expressions for p(κ) have the form:

$$p(\varkappa)=V(\varkappa)f^{-2}(\varkappa)\exp\left[i2(\omega_2-u_H)-i\frac{5\pi}{6}\right], \quad |\arg u_H|\leqslant\frac{\pi}{2}; \tag{4a}$$

$$p(\varkappa)=\widetilde{V}(\varkappa)\exp\left(i2\omega_2-i\frac{\pi}{3}\right)\left[1-f^{2}(\varkappa)\exp\left(-i2u_H+\frac{i\pi}{2}\right)\right], \quad \frac{\pi}{2}\leqslant\arg u_H\leqslant\frac{3\pi}{2}, \tag{4b}$$

where

$$V(\varkappa)=\frac{\rho-\rho_g}{\rho+\rho_g}+\frac{i}{3\eta_H}\left(1+\frac{\rho}{\rho_g}\right)^{-1}\left(1+0\left(\frac{1}{u_H}\right)\right);$$

$$\widetilde{V}(\varkappa)=\left(1+\frac{\rho_g}{\rho}\right)^{-1}\left[1-\frac{\rho_g}{\rho}\cdot\frac{1+f^2(\varkappa)\exp\left(i2u_H-i\frac{\pi}{2}\right)}{1-f^2(\varkappa)\exp\left(i2u_H-i\frac{\pi}{2}\right)}\right]+$$
$$+i\left\{3\eta_H\left(1+\frac{\rho}{\rho_g}\right)\left[1-f^2(\varkappa)\exp\left(i2u_H-i\frac{\pi}{2}\right)\right]\right\}^{-1}\left[1+0\left(\frac{1}{u_H}\right)\right];$$

ρ is the density of the medium at $h_0 \leq z \leq H$; ρ_g is the density of the medium at

$$z\geqslant H;\quad u_H=u(H;\varkappa);\quad u(z;\varkappa)=k_0\int_z^{z_2}\sqrt{n^2(\xi)-\varkappa^2}\,d\xi;$$

arg u(z, H) on the upper sheet of the Riemann surface is set by the same conditions as for ω(z; κ). In the derivation of formulas (4a) and (4b) it was assumed that $2b = |\frac{d}{dz}n^2(z)|$.

It is easy to see that the right side of equation (1) is identically equal to the right side of the equality

$$\Psi=(r,z,z_0)=\sum_{N=0}^{N_0-1}\Psi_N(r,z,z_0)+\widetilde{\Psi}_{N_0}(r,z,z_0), \tag{5}$$

where

$$\Psi_N(r,z,z_0)=\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon}F_N(r,z,z_0;\varkappa)\,d\varkappa; \tag{6a}$$

$$\widetilde{\Psi}_{N_0}(r,z,z_0)=\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon}F_{N_0}(r,z,z_0;\varkappa)\,[1-p(\varkappa)q(\varkappa)]^{-1}\,d\varkappa. \tag{6b}$$

Here

$$F_0(r, z, z_0; \varkappa) = \frac{i\pi k_0^2\varkappa}{4}\Phi_1(z_0; \varkappa)\left[\Phi_2(z; \varkappa) - q(\varkappa)\Phi_1(z; \varkappa)\right] H_0^1(k_0 r\varkappa);$$
$$F_N(r, z, z_0; \varkappa) = -i\frac{\pi k_0^2\varkappa}{4}\left[\Phi_2(z_0; \varkappa) - q(\varkappa)\Phi_1(z_0; \varkappa)\right] \times$$
$$\times\left[\Phi_2(z; \varkappa) - q(\varkappa)\Phi_1(z; \varkappa)\right] q^{N-1}\varkappa p^N(\varkappa) H_0^{(1)}(k_0 r\varkappa), \quad N \geqslant 1. \tag{7}$$

The representation of expression (5) is very convenient for calculating the field in a region where there is a zonal structure; calculation according to formula (5) is simpler than a corresponding calculation by the normal-wave method if the number of normal waves (nondamping from r) is large.

Formula (5) permits us to calculate comparatively easily the acoustic field in the zones of the refraction geometric shadow. The specific form of the expressions obtained as a result of calculating integrals (6a, 6b) depends on both the distribution of $n^2(z)$ and the position of the layer boundaries, as well as on the location of the sound source and the point of observation.

Let us examine the expression for the acoustic field in the first zone of the refraction geometric shadow under the condition that

$$n^2(z) = 2\exp[a(h-z)] - \exp[2a(h-z)],\ ak_0(h-h_0)^2 \gg 1.$$

Let us note that in this case the solution to equation (2) can be expressed through special investigated functions so that

$$\Phi_1(z; \varkappa) = -\frac{1}{\pi}k_0^{-1/2}\left[\Gamma\left(\lambda+\mu+\frac{1}{2}\right)\Gamma\left(\lambda-\mu+\frac{1}{2}\right)\right]^{1/2}\times$$
$$\times e^{i\pi\lambda + i\pi/3 + a/2(z-h)} W_{-\lambda,\mu}(\xi e^{i\pi});$$
$$\Phi_2(z; \varkappa) = -\frac{1}{\pi}k_0^{-1/2}\left[\Gamma\left(\lambda+\mu+\frac{1}{2}\right)\Gamma\left(\lambda-\mu+\frac{1}{2}\right)\right]^{1/2}\times$$
$$\times e^{-i\pi\lambda - i\pi/3 + a/2(z-h)} W_{-\lambda,\mu}(\xi e^{-i\pi}), \tag{8}$$

where

$$\lambda = \frac{k_0}{a};\quad \mu = \frac{k_0\varkappa}{a};\quad \xi = \xi(z) = 2\frac{k_0}{a}\exp[a(h-z)].$$

Having solved equation (6a) by the residue method we get

$$\Psi_0(r,\ z,\ z_0)=\sum_{l=1}^{\infty} A_{0,l}(r,\ z,\ z_0)\exp\left[-k_0(r-R_{00})\operatorname{Im}\varkappa_l\right], \tag{9}$$

where

$$\Phi_1(h_0;\ \varkappa_l)=0,\quad 0<\arg\varkappa_l<\frac{\pi}{2};$$

$$A_{0,l}=\sqrt{\frac{k_0\varkappa_l}{2\pi r}}\cdot\frac{4\pi\sin\left(\omega_{0,l}-\frac{\pi}{4}\right)\sin\left(\omega_l-\frac{\pi}{4}\right)}{\sqrt[4]{n^2(z_0)-\varkappa_l^2}\,\sqrt[4]{n^2(z)-\varkappa_l^2}\left[\frac{\partial}{\partial\varkappa}\omega(h_0\varkappa)\right]_{\varkappa=\varkappa_l}}\times$$
$$\times\exp\left[ik_0R_{00}\varkappa_l+ik_0(r-R_{00})\operatorname{Re}\varkappa_l+i\frac{\pi}{4}\right](1+Q_l). \tag{10a}$$

If $(z - h_0)$ and $(z_0 - h_0)$ are high and ℓ is low,

$$A_{0,l}\simeq\sqrt{\frac{k_0\varkappa_l}{2\pi r}}\cdot\frac{\pi\exp(i\varphi_{0,l})}{\left[\frac{\partial}{\partial\varkappa}\omega(h_0,\ \varkappa)\right]_{\varkappa=\varkappa_l}}\cdot\frac{\exp\left\{\frac{k_0}{2}\left(\frac{\partial}{\partial\varkappa}R_{00}\right)\operatorname{Im}\left[\varkappa_l-n(h_0)\right]^2\right\}}{\sqrt[4]{n^2(z_0)-\varkappa_l^2}\,\sqrt[4]{n^2(z_0)-\varkappa_l^2}}. \tag{10b}$$

In equations (10a) and (10b) $z_0 < h$, $z < h$, $\omega_{0,\ell} = \omega(z_0, \kappa_\ell)$, $\omega_\ell = \omega(z, \kappa_\ell)$, $|Q_\ell| << 1$, where

$$\omega(z;\ \varkappa)=k_0\int_{z_1}^{z}\left[n^2(\xi)-\varkappa^2\right]^{1/2}d\xi=\frac{k_0}{a}\left\{\arcsin\sqrt{\frac{n^2(z)-\varkappa^2}{1-\varkappa^2}}-\right.$$
$$\left.-\varkappa\arcsin\varkappa e^{a(z-h)}\sqrt{\frac{n^2(z)-\varkappa^2}{1-\varkappa^2}}-\sqrt{n^2(z)-\varkappa^2}\right\};$$
$$\frac{\partial}{\partial\varkappa}\omega(h_0;\ \varkappa)=-\frac{k_0}{a}\arcsin\left(\varkappa e^{a(z-h)}\frac{\sqrt{n^2(h_0)-\varkappa^2}}{\sqrt{1-\varkappa^2}}\right);$$
$$R_{00}=R_0(z,\ z_0;\ n(h_0));$$
$$R_0(z,\ z_0;\ \varkappa)=\varkappa\int_{z_1}^{z_0}\left[n^2(\xi)-\varkappa^2\right]^{-1/2}d\xi+\varkappa\int_{z_1}^{z}\left[n^2(\xi)-\varkappa^2\right]^{-1/2}d\xi, \tag{11}$$

where

$$\varkappa\int_{z_1}^{z}\left[n^2(\xi)-\varkappa^2\right]^{-1/2}d\xi=\frac{\pi}{2a}-\frac{1}{a}\arcsin\left\{1-\frac{\varkappa^2\exp\left[a(z-h)\right]}{\sqrt{1-\varkappa^2}}\right\};$$
$$\varphi_{0,l}=k_0n(h_0)r+\omega\left[z_0;\ n(h_0)\right]+\omega\left[z;\ n(h_0)\right]+k_0(r-R_{00})\times$$
$$\times\left[\operatorname{Re}\varkappa_l-n(h_0)\right]-\frac{k_0}{2}\left(\frac{\partial}{\partial\varkappa}R_{00}\right)\operatorname{Re}\left[\varkappa_l-n(h_0)\right]^2;$$
$$\frac{\partial}{\partial\varkappa}R_{00}=\left[\frac{\partial}{\partial\varkappa}R_0(z_1,\ z_0;\ \varkappa)\right]_{\varkappa=n(h_0)}.$$

In accordance with the geometric theory for low values of $(z - h_0)$ the boundary between the shadow zone and the first illuminated zone is formed by a beam which has touched the surface $z = h_0$, and the distance to the shadow zone boundary is equal to R_{00}. In this case $((\partial/\partial\kappa)R_{00}) < 0$ and $|A_{0,\ell}|$ decreases with increasing ℓ. As a result of this, at $r \geq R_{00}$ in formula (9) we may restrict ourselves to the first terms of the series.

If the value $(z - h_0)$ is high, the shadow zone boundary is formed by a caustic. On the caustic $r = R_{0,k} = R_0(z, z_0; \kappa_{0,k})$; $\kappa_{0,k}$ is a root of the equation $(\partial/\partial\kappa)R_0(z, z_0; \kappa) = 0$.

In such a case $((\partial/\partial\kappa)R_{00}) > 0$, and according to formula (10b) value $A_{0,\ell}$ initially increases with increasing ℓ and only at sufficiently high values of ℓ does it begin to decrease. Therefore, in the vicinity of the shadow zone boundary it becomes necessary to take the large number of terms of the series of equation (9) into account.

Now we get an expression for $\Psi_0(r, z, z_0)$ convenient for calculation in this case, and we deform the integration contour into a loop encompassing the poles of function $q(\kappa)$ in integral formula (6a). On this contour

$$\Psi_0(r, z, z_0) = \int\limits_{c_0} \sqrt{\frac{k_0\kappa}{2\pi r}} \cdot \frac{f^2(\kappa)\, q(\kappa)}{\sqrt[4]{n^2(z_0) - \kappa^2}\,\sqrt[4]{n^2(z) - \kappa^2}} \times \times e^{i\varphi_0(r, z, z_0; \kappa) + i5\pi/12}(1 + \Lambda_0(\kappa)), \tag{12}$$

where

$$\Lambda_0(\kappa) = 0\left(\frac{1}{\omega}, \frac{1}{\omega_0}, \frac{1}{k_0 r\kappa}\right); \quad |\Lambda_0(\kappa)| \ll 1$$

at $|\omega| > 1, \; |\omega_0| > 1, \; |k_0 r\kappa| \gg 1$;

$\varphi_0(r, z, z_0; \kappa) = k_0 r\kappa + \omega_0 + \omega$ is the phase.

If the shadow zone boundary is formed by the caustic, the phase $\phi_0(r, z, z_0)$ has its saddle point κ_0 in which $1 - \kappa^2 = \rho e^{i\Phi}$, where $\Phi \leq 0$, if $r \geq R_{0,k}$; $\kappa_0 = \kappa_{0,k}$, if $r = R_{0,k}$. Value κ is the root of the equation

$$\frac{1}{k_0}\frac{\partial}{\partial\kappa}\varphi_0(r, z, z_0, \kappa) = r - R_0(z, z_0, \kappa) = 0,$$

where $R_0(z, z_0; \kappa)$ is set by the equality of (11).

If $z = z_0$, $\rho = \exp[a(h - z) - 1$;

$$R_0(z_0, z_0; \varkappa_0) = \frac{\pi}{a} - \frac{2}{a}\arcsin 2e^{a(z-h)}\sqrt{e^{a(h-z)} - 1}\cos\frac{\Phi}{2}.$$

If $0 \geq \Phi > \pi$, then $\mathrm{Im}\ \kappa_0 \geq 0$; $f(\kappa_0) \approx 1$. Let us plot the integration contour in formula (12) through the point $\kappa = \kappa_0$ along the line of the most rapid decrease (with an increase of $|\kappa - \kappa_0|$) of the factor $\exp[i\phi_0(r, z, z_0; \kappa)]$. Assuming that $|\phi_0(\kappa_0) - \phi_0(\kappa_e)| \gg 1$ and $|\Lambda_0(\kappa_0)| \ll 1$, we get

$$\Psi_0(r, z, z_0) = \Psi_{0,k}(r, z, z_0) + \sum_l \Psi_{0,l}(r, z, z_0), \tag{13}$$

where summation is performed according to the numbers ℓ of the poles touched in the deformation of the contour, and

$$\begin{aligned}\Psi_{0,k}(r, z, z_0) = {} & \frac{\varkappa_0^{1/2} 2^{5/6} k_0^{1/6} f^2(\varkappa_0) v(t)(1 + Q_{0,k})}{r^{1/2}\left|n^2(z_0) - \varkappa_0^2\right|^{1/4}\left|n^2(z) - \varkappa_0^2\right|^{1/4}} \times \\ & \times \exp\left\{i\varphi_0(r, z, z_0; \varkappa) - \frac{ik_0}{3}\left[\frac{\partial}{\partial\varkappa} R_0(z, z_0; \varkappa)\right]^3_{\varkappa=\varkappa_0} \times \right. \\ & \times \frac{\left.\times\left[-\frac{\partial^2}{\partial\varkappa^2} R_0(z, z_0; \varkappa)_{\varkappa=\varkappa_0}\right]^2 - \frac{i\pi}{4}\right\}}{\left[-\frac{\partial^2}{\partial\varkappa^2} R_0(z, z_0; \varkappa)_{\varkappa=\varkappa_{0,k}}\right]^{1/3}},\end{aligned} \tag{14}$$

where $v(t)$ is the Airy function;

$$t = \left(\frac{k_0}{2}\right)^{2/3}\left[\frac{\partial}{\partial\varkappa} R_0(z, z_0; \varkappa)_{\varkappa=\varkappa_0}\right]^2\left[-\frac{\partial^2}{\partial\varkappa^2} R_0(z, z_0; \varkappa)\right]^{-4/3}_{\varkappa=\varkappa_0}; \quad r \geqslant R_{0,k};$$

$$|Q_{0,k}| \ll 1 \quad \text{at} \quad |\Lambda_0(\varkappa)_0| \ll 1, \quad |w(h_0; \varkappa_0)| > 1,$$

$$|\varkappa_l - \varkappa_0| \geqslant \pi k_0^{-1/3}\left|\frac{\partial}{\partial\varkappa} R_0(z, z_0; \varkappa)\right|^{-1/3}_{\varkappa=\varkappa_0}.$$

Function $\Psi_{0,k}(r, z, z_0)$ at $r > R_{0,k}$ describes the diffraction from the caustic. For an unbound medium $\lim \Psi_0(r, z, z_0) = \Psi_{0,k}(r, z, z_0)$.

Surface $z = h_0$ places limitations on the geometric dimensions of the caustic and introduces perturbations into the field diffracted from it. The series in ℓ in equation (13) considers this perturbation which depends both on the position and on the reflection properties of the upper boundary. At low values of t we have

$$\Psi_0(r,\ z,\ z_0) \simeq \Psi_{0,1}(r,\ z,\ z_0) \simeq \Psi_{0,k}(r,\ z,\ z_0).$$

The series in ℓ in formula (13) can also make a noticeable contribution to $\Psi_0(r, z, z_0)$, if $t \gg 1$; in this case $\Psi_{0,k}(r, z, z_0) \lesssim \Psi_{0,1}(r, z, z_0)$ and $\Psi_0(r, z, t_0) \simeq \Psi_{0,1}(r, z, z_0) + \Psi_{0,k}(r, z, z_0)$.

Let us now examine the contribution to $\Psi(r, z, z_0)$ of other components which enter the sum of expression (5) under the condition $h_0 = -\infty$.

Let us assume that $n(z_0) > n(H)$ (the beam picture for this case is depicted in Fig. 1). In accordance with geometric theory the first zone of the refraction shadow is limited on both sides by illuminated zones when $n(z) > n(H)$. If $n(z) < n(H)$, horizon z intersects only the first illuminated zone and does not pass through the second and subsequent illuminated zones, so that the shadow zone is unbound in the positive direction of axis r.

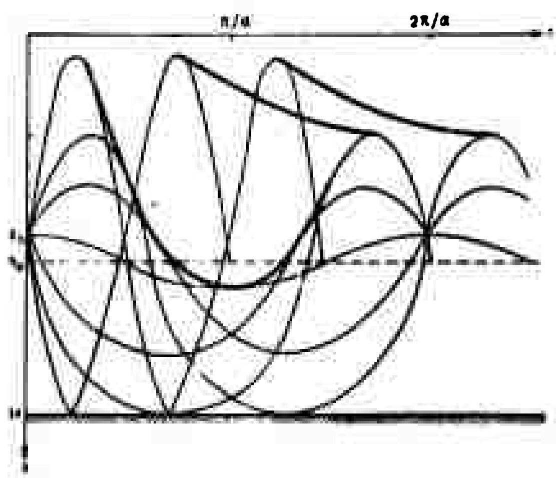

Fig. 1. Beam picture (heavy lines – the caustic).

Let us examine the most interesting case when $n(z_0) > n(H)$ and $n(z) > n(H)$; we assume that distances $(h - z_0)$ and $(h - z)$ are sufficiently small so that the boundary between the shadow zone and

the secondary illuminated zone is formed by the caustic, the distance to which let us designate by $R_{1,k}$. In accordance with geometric theory, this caustic is the envelope of a family of beams whose equation has the form

$$r = R_{1,1}(z, z_0; \varkappa_{1,1}), \tag{15}$$

where $\kappa_{1,1} = n(z_0) \cos \alpha$ is a real number and α is the angle of beam emergence from the source.

$$R_{1,1}(z, z_0; \varkappa) = \varkappa \int_{z_0}^{z_1} [n^2(\xi) - \varkappa^2]^{-1/2} d\xi + \varkappa \int_{z}^{z_1} [n^2(\xi) - \varkappa^2]^{-1/2} d\xi.$$

After substituting n(z) into the latter expression, we get

$$\varkappa \int_{z}^{z_1} [n^2(\xi) - \varkappa^2]^{1/2} d\xi = \frac{\pi}{2a} + \frac{1}{a} \arcsin\left(\frac{1 - \varkappa^2 \exp[a(z-h)]}{\sqrt{1-\varkappa^2}}\right).$$

The position of the point on the caustic is set by the equation

$$r = R_{1,1}(z, z_0; \varkappa_{1,k}),$$

where $\kappa_{1,k}$ is the root of the equation

$$\frac{\partial}{\partial \varkappa} R_{1,1}(z, z_0; \varkappa) = 0.$$

Substituting equations (3) and (4) into formula (7), for $N = 1$, $|k_0 r \kappa| >> 1$, $|f(n(H))| \simeq 1$ we get

$$F_1(r, z, z_0; \varkappa) = \sqrt{\frac{k_0 \varkappa}{2\pi r}} \frac{V(\varkappa)(1 + \Delta(z_0))(1 + \Delta(z))}{\sqrt{n^2(z_0) - \varkappa^2}\sqrt{n^2(z) - \varkappa^2}\, f(\varkappa)} \times$$
$$\times [1 + f^2(\varkappa) e^{-i2u_H + i\pi/2}][1 + \Delta_{1,1}(\varkappa)] \times \exp[i\varphi_{1,1}(r, z, z_0; \varkappa)] \equiv$$
$$\equiv F_{1,1}(r, z, z_0; \varkappa), \tag{16a}$$

if $\pi/2 \leq \arg u_H \leq 3\pi/2$;

$$\varphi_{1,1}(r, z, z_0; \varkappa) = k_0 r \varkappa + w(z_0; \varkappa) + 2w_2 + w(z, \varkappa) - \frac{\pi}{4}; \quad w_2 = \varkappa k_0 a^{-1}(1 - \varkappa);$$

$$\Delta(z) = \begin{cases} 0, & \text{if } \frac{\pi}{2} \leq \arg w < \frac{3\pi}{2} \\ f^2(\varkappa) \exp\left[i2w - \frac{i\pi}{2}\right], & \text{if } |\arg w| < \frac{\pi}{2} \end{cases}$$

$$\Lambda_{1,1}(\varkappa) = O\left(\frac{1}{w}, \frac{1}{w_0}, \frac{1}{u_H}, \frac{1}{k_0 r \varkappa}\right); |\Lambda_{1,1}(\varkappa)| \ll 1, \text{ if } |w_0| > 1, |w| > 1, |u_H| > 1, |k_0 r \varkappa| > 1.$$

If $|\arg u_H| \leq \pi/2$, however, then

$$F_1(r, z, z_0; \varkappa) = \sum_{j=2}^{5} \sqrt{\frac{k_0 \varkappa}{2\pi r}} \cdot \frac{V(\varkappa)\left(1 + \Lambda_{1,j}^{(\varkappa)}\right)}{\sqrt[4]{n^2(z_0) - \varkappa^2}\sqrt[4]{n^2(z) - \varkappa^2}} \times$$
$$\times e^{-i\varphi_{1,j}(r, z, z_0; \varkappa)} = \sum_{j=2}^{5} F_{1,j}(r, z, z_0; \varkappa), \tag{16b}$$

where

$$\varphi_{1,2}(r, z, z_0; \varkappa) = k_0 r \varkappa + k_0 \int_{z_0}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi + k_0 \int_{z}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi - \frac{3\pi}{4};$$

$$\varphi_{1,3}(r, z, z_0; \varkappa) = k_0 r \varkappa + k_0 \int_{z_0}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi +$$
$$+ k_0 \int_{z_1}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi + k_0 \int_{z_1}^{z} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi - \frac{5\pi}{4};$$

$$\varphi_{1,4}(r, z, z_0; \varkappa) = k_0 r \varkappa + k_0 \int_{z_1}^{z} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi +$$
$$+ k_0 \int_{z_1}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi + k_0 \int_{z}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi - \frac{5\pi}{4};$$

$$\varphi_{1,5}(r, z, z_0; \varkappa) = k_0 r \varkappa + k_0 \int_{z_1}^{z_0} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi +$$
$$+ 2k_0 \int_{z_1}^{H} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi + k_0 \int_{z_1}^{z} \sqrt{n^2(\xi) - \varkappa^2}\, d\xi + \frac{\pi}{4};$$

$$|\Lambda_{1,j}(\varkappa)| \ll 1, \text{where } |w_0| > 1, |w| > 1, |u_H| > 1, |k_0 r \varkappa| \gg 1.$$

Let us assume that $r \leq R_{1,k}$; then functions $\phi_{1,j}(r, z, z_0; \kappa)$, $j = 1, 2, 3, 4, 5$ have saddle points $\kappa_{1,j}$, located in the region of applicability of the representations of formulas (16a) and (16b). We deform the integration contour of the integral of equation (6a) described by $N = 1\Psi_1(r, z, z_0)$ so that it passes these saddle points along corresponding saddle-point contours, while outside the essential saddle-point vicinities it goes along the lines of most rapid decrease $F_1(r, z, z_0; \kappa)$. The integration contour in

region $|\arg u_H| \leq \pi$ is split into four branches by the four respective components in expression (16b); these branches merge at the origin of $p(\kappa)$. Stemming from this, the integration contour will pass further through saddle point $\kappa_{1,1}$, $\mathrm{Im}\,\kappa_{1,1} < 0$ of function $\Phi_{1,1}(r, z, z_0; \kappa)$ (Fig. 2). Saddle point $\kappa_{1,j}$ satisfies the equation $(\partial/\partial\kappa)\Phi_{1,j}(r, z, z_0; \kappa)$, which is reduced to the form

$$r = R_{1,j}(z, z_0; \varkappa), \tag{17}$$

whereupon function $R_{1,1}(z, z_0; \kappa)$ coincides with function (15):

$$\begin{gathered}
R_{1,j}(z, z_0; \varkappa) = \varkappa \int_{z_0}^{H} |n^2(\zeta) - \varkappa^2|^{-1/2} d\zeta + \varkappa \int_{z_1}^{H} |n^2(\zeta) - \varkappa^2|^{-1/2} d\zeta + \\
+ (-1)^{j-1} \varkappa \int_{z_1}^{z} |n^2(\zeta) - \varkappa^2|^{-1/2} d\zeta, \\
j = 2, 3; \\
R_{1,j}(z, z_0; \varkappa) = \varkappa \int_{z_1}^{z_0} |n^2(\zeta) - \varkappa^2|^{-1/2} d\zeta + \\
+ 2\varkappa \int_{z_1}^{H} |n^2(\zeta) - \varkappa^2|^{-1/2} d\zeta + (-1)^{j-1} \varkappa \int_{z_1}^{z} |n^2(\zeta) - \varkappa^2|^{-1/2} d\zeta, \\
j = 4, 5.
\end{gathered} \tag{18}$$

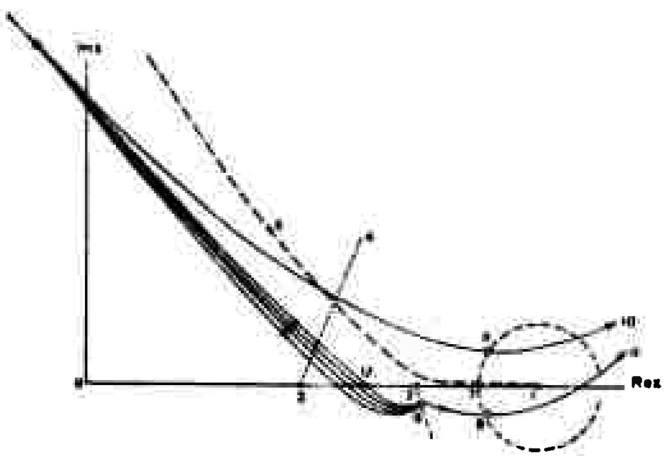

Fig. 2. Plane κ.
1 – Point $n(0) = 1$; 2 – point $n(H)$; 3 – point $n(h_0)$; 4 – origin of function $\Phi_1(h_0; \kappa)$; 5 – pole of the integrand in formula (1); 6 – origin of function $p(\kappa)$; 7 – point $\kappa_{0,k}$ and point $\kappa_{1,k}$; 8 – point κ_0; 9 – point $\kappa_{1,1}$; 10 – integration contour C_0; 11 – integration

contour C_1; 12 – family of integration contours C_j, $j = 2, 3, 4, 5$.

From equation (17) it follows that $0 < \kappa_{1,j} < n(H)$ and $\mathrm{Im}\,\kappa_{1,j} = 0$ for $j = 2, 3, 4, 5$. Saddle point $\kappa_{1,1}$ at $r < R_{1,k}$ lies in half-plane $\mathrm{Im}\,\kappa < 0$ on the line $1 - \kappa_{1,1}^2 = \rho e^{i\Phi}$, $0 \le \Phi < \pi$. At $z = z_0$ $\rho = \exp[a(h - z_0)] - 1$.

Correspondingly, for $R_{1,1}(z_0, z_0; \kappa_{1,1})$ we have

$$R_{1,1}(z_0, z_0; \varkappa_{1,1}) = \frac{\pi}{a} + \frac{2}{a}\arcsin 2e^{a(z_0 - h)}\sqrt{e^{a(h - z_0)} - 1}\cos\frac{\Phi}{2}.$$

Calculating the integral of expression (6a) according to the new integration contour at $r \le R_{1,k}$ we get

$$\Psi_1(r, z, z_0) = \sum_{j=1}^{5} \Psi_{1,j}(r, z, z_0), \tag{19}$$

if $j = 2, 3, 4, 5$, then

$$\Psi_{1,j}(r, z, z_0) = \sqrt{\frac{\varkappa_{1,j}}{r\left[\frac{\partial}{\partial\varkappa} R_{1,j}(z, z_0; \varkappa)_{\varkappa=\varkappa_{1,j}}\right]}} \times \\ \times \frac{V(\varkappa_{1,j})\exp\left[i\varphi_{1,j}(r, z, z_0; \varkappa_{1,j}) - \frac{i\pi}{4}\right]}{\sqrt[4]{n^2(z_0) - \varkappa_{1,j}^2}\,\sqrt[4]{n(z) - \varkappa_{1,j}^2}}(1 + Q_{1,j}), \tag{19a}$$

where $|Q_{1,j}| \ll 1$ at $|\Lambda_{1,j}(\kappa_{1,j})| \ll 1$;

$$\Psi_{1,1}(r, z, z_0) = \frac{k_0^{1/6} 2^{5/6} \varkappa_{1,1}^{1/2}}{r^{1/2}\left[\frac{\partial^2}{\partial\varkappa^2} R_{1,1}(z, z_0, \varkappa)_{\varkappa=\varkappa_{1,1}}\right]^{1/3}} \times \\ \times \frac{V(t)(1 + Q_{1,1})}{f^2(\varkappa_{1,1})\sqrt[4]{n^2(z_0) - \varkappa_{1,1}^2}\,\sqrt[4]{n^2(z) - \varkappa^2}} \exp\Big\{ i\varphi_{1,1}(r, z, z_0; \varkappa_{1,1}) - \\ - \frac{ik_0}{3}\left[\frac{\partial}{\partial\varkappa} R_{1,1}(z, z_0; \varkappa)_{\varkappa=\varkappa_{1,1}}\right]^3\left[\frac{\partial^2}{\partial\varkappa^2} R_{1,1}(z, z_0; \varkappa)_{\varkappa=\varkappa_{1,1}}\right]^2\Big\}. \tag{19b}$$

Here

$$t = e^{i\pi} k_0^{2/3} \cdot 2^{-2/3}\left[\frac{\partial}{\partial\varkappa} R_{1,1}(z, z_0; \varkappa)_{\varkappa=\varkappa_{1,1}}\right]^2\left[\frac{\partial^2}{\partial\varkappa^2} R_{1,1}(z, z_0; \varkappa)_{\varkappa=\varkappa_{1,1}}\right]^{-4/3};$$

$|Q_{1,1}| \ll 1$, if $|\Lambda_{1,1}(\kappa_{1,1})| \ll 1$.

If $\Phi \ll 1$ (small values of $R_{1,k} - r$),

$$R_{1,1}(z, z_0;\ \varkappa_{1,1}) \simeq R_{1,k} + \frac{1}{2}\left[\frac{\partial^2}{\partial\varkappa^2} R_{1,1}(z, z_0;\ \varkappa)_{\varkappa=\varkappa_{1,k}}\right](\varkappa_{1,1} - \varkappa_{1,k})^2 \text{ and}$$

$$t \simeq 2^{1/3} k_0^{2/3} (R_{1,k} - r)\left[\frac{\partial^2}{\partial\varkappa^2} R_{1,1}(z, z_0;\ \varkappa)_{\varkappa=\varkappa_{1,k}}\right]^{-1/3}.$$

In this case $\Psi_{1,1}(r, z, z_0)$ is reduced to the form

$$\Psi_{1,1}(r, z, z_0) = \\ = \frac{k_0^{1/6} \cdot 2^{5/6} \varkappa_{1,k} V(t)(1 + Q_{1,1})}{r^{1/2}\left[\frac{\partial^2}{\partial\varkappa^2} R_{1,1}(z, z_0;\ \varkappa)_{\varkappa=\varkappa_{1,k}}\right]^{1/3} \sqrt[4]{n^2(z_0) - \varkappa_{1,k}^2}\ \sqrt[4]{n^2(z) - \varkappa_{1,k}^2}} \times \\ \times e^{i\varphi_{1,1}(r, z, z_0;\ \varkappa_{1,k})}. \qquad (19c)$$

Formula (19c) describes the acoustic field in the region of the caustic. It is valid [1] not only with $r < R_{1,k}$, when $t > 0$, but also with $r > R_{1,k}$, if $t \gtrsim 1$. If $t \gg 1$,

$$\Psi_{1,1}(r, z, z_0) = \sqrt{\frac{\varkappa_{1,1}}{r\left[\frac{\partial}{\partial\varkappa} R_{1,1}(z, z_0;\ \varkappa)_{\varkappa=\varkappa_{1,1}}\right]}} \times \\ \times \frac{\exp\left[i\varphi_{1,1}(r, z, z_0;\ \varkappa_{1,1}) - \frac{i\pi}{4}\right]}{f^2(\varkappa_{1,1})\sqrt[4]{n^2(z_0) - \varkappa_{1,1}^2}\ \sqrt[4]{n^2(z) - \varkappa_{1,1}^2}}(1 + Q_{1,1}), \qquad (19a)$$

whereupon

$$\varphi_{1,1}(r, z, z_0;\ \varkappa_{1,1}) = \frac{k_0\pi}{a} - \frac{k_0}{a}\arcsin\left\{\frac{1 - \exp[a(h - z_0)]}{\sqrt{\rho}} e^{-i\Phi/2}\right\} - \\ - \frac{k_0}{a}\arcsin\left\{\frac{1 - \exp[a(h - z)]}{\sqrt{\rho}} e^{-i\Phi/2}\right\} + \frac{k_0}{a}\sqrt{\rho e^{i\Phi} - (e^{a(h - z_0)} - 1)^2} + \\ + \frac{k_0}{a}\sqrt{\rho e^{i\Phi} - (e^{a(h - z)} - 1)^2}.$$

It is essential to note that the use of formula (19c) at $t \gg 1$ (far from the caustic) gives greatly reduced values for the function $\Psi_{1,1}(r, z, z_0)$ as compared to the values calculated from exact formula (19b).

Let us note that $|\Psi_{1,1}(r, z, z_0)|$ monotonically diminishes with an increase of $(R_{1,k} - r)$.

The integral of formula (6a) at $N = 1$ also permits another evaluation, convenient at high values of $(R_{1,k} - r)$. Let us set the value of $r = r_1$ to which corresponds $\varkappa_{1,1}$, which satisfies the condition $0 < \Phi << \pi/2$. In this case we can plot the integration contour in equation (6a) through $\varkappa_{1,1}$ without removing the integral from the convergence regions such that $\mathrm{Im}\,\varkappa \geq \mathrm{Im}\,\varkappa_{1,1}$ on it. Then we get

$$|\Psi_{1,1}(r, z, z_0)| \leqslant \int_{C_1} |F_{1,1}(r, z, z_0; \varkappa)||d\varkappa| \leqslant e^{-k_0(r_1 - r)|\mathrm{Im}\,\varkappa_{1,1}|} \times \\ \times \int_{C_1} |F_{1,1}(r_1, z, z_0; \varkappa)||d\varkappa|,$$

whence it follows that at $r \leq r_1$

$$|\Psi_{1,1}(r, z, z_0)| \leqslant |\Psi_{1,1}(r_1, z_0, z)| e^{-k_0(r_1 - r)|\mathrm{Im}\,\varkappa_{1,1}|}. \tag{20}$$

In accordance with expression (5) $\Psi(r, z, z_0) = \Psi_0(r, z, z_0) + \tilde{\Psi}_1(r, z, z_0)$. Comparing formulas (6a) and (6b) at $N = N_0 = 1$, we see that the integral expression for $\tilde{\Psi}_1(r, z, z_0)$ differs from the corresponding expression for $\tilde{\Psi}_1(r, z, z_0)$ only by the factor $[1 - p(\varkappa)q(\varkappa)]^{-1}$ in the integrand. Additional peculiarities of the pole type introduced by this factor into the integrand for $\tilde{\Psi}_1(r, z, z_0)$ lie partially on the real $\varkappa$ axis at $1 > \varkappa^2 > n^2(H)$, and partially in the regions: $n(H) \gtrsim \mathrm{Re}(\varkappa) \geq 0$, $\mathrm{Im}\,\varkappa > 0$ and $0 \geq \mathrm{Re}\,\varkappa \geq -n(H)$, $\mathrm{Im}\,\varkappa < 0$ (see Fig. 2).

If $r < R_{1,k}$, the integration contour in the integral of formula (6b) at $N = 1$ can be deformed into exactly the same integration path which was used in the calculation of $\Psi_1(r, z_0, z)$. With such transformation of the integration contour the poles of function $[1 - p(\varkappa)q(\varkappa)]^{-1}$ are not affected and, in addition, the new contour runs in the same part of plane $\varkappa$ in which $p(\varkappa)q(\varkappa)| < 1$, if $|V(\varkappa)| < 1$. After deforming the integration contour into contour C_j^1 by the method shown and after estimating the value of the integral we get

$$\tilde{\Psi}_1(r, z, z_0) = \sum_{j=1}^{\tilde{j}} \Psi_{1,j}(r, z, z_0) = \sum_{j=1}^{\tilde{j}} \Psi_{1,j}(r, z, z_0)\left(1 + \tilde{Q}_{1,j}\right). \tag{21}$$

where

$$|\tilde{Q}_{1,j}| < \max \left| \frac{p(\varkappa) q(\varkappa)}{1 - p(\varkappa) q(\varkappa)} \right|_{\varkappa \in C_j}.$$

In other words,

$$|\tilde{Q}_{1,1}| < \left| \exp\left[2^{3/2} \pi k_0 a^{-1} (R_{1,*} - r)^{1/2} \left(\frac{d^2}{d\varkappa^2} R_{1,1}(z, z_0; \varkappa)_{\varkappa = \varkappa_{1,*}}\right)^{-1/2}\right] - 1 \right|^{-1}; \tag{22a}$$

$$|\tilde{Q}_{1,j}| \leqslant \frac{|V(\varkappa_{1,j})|}{1 - |V(\varkappa_{1,j})|}, \quad j = 2, 3, 4, 5. \tag{22b}$$

Estimation of equations (22b) is convenient if $|V(\kappa_{1,j})| \ll 1$. The expression for $\tilde{\Psi}_{1,j}(r, z, z_0)$, $j = 2, 3, 4, 5$ can also be obtained in the form

$$\tilde{\Psi}_{1,j}(r, z, z_0) = \sum_{N=1}^{\infty} \Psi_{N,j}(r, z, z_0), \tag{23}$$

where

$$\Psi_{N,j}(r, z, z_0) = \sqrt{\frac{\varkappa_{N,j}}{r\left(\frac{d}{d\varkappa} R_{N,j}(z, z_0; \varkappa)\right)}} \times$$
$$\times \frac{V^N(\varkappa_{N,j}) \exp\left[i\varphi_{N,j}(r, z, z_0; \varkappa_{N,j}) + \frac{i\pi}{2}(N-1) - \frac{i\pi}{4}\right]}{\sqrt[4]{n^2(z_0) - \varkappa_{N,j}^2}\sqrt[4]{n^2(z) - \varkappa_{N,j}^2}} (1 + Q_{N,j}). \tag{24}$$

Here

$$\varphi_{N,j}(r, z, z_0; \varkappa) = \varphi_{1,j}(r, z, z_0; \varkappa) + 2(N-1)w(H, \varkappa);$$
$$R_{N,j}(z, z_0; \varkappa) = R_{1,j}(z, z_0; \varkappa) + 2(N-1)\varkappa \int_{z_t}^{H} [n^2(\xi) - \varkappa^2]^{-1/2} d\xi;$$
$$|Q_{N,j}| \ll 1, \quad |u_H| > 1, \quad |k_0 r \varkappa| \gg 1; \tag{25}$$

$\kappa_{N,j}$ is the root of the equation $r = R_{N,j}(z, z_0; \kappa)$.

Formula (23) is obtained by replacing $[1 - p(\kappa)q(\kappa)]^{-1}$ by $\sum_{N=1}^{\infty} [p(\kappa)q(\kappa)]^N$ in the integral describing $\tilde{\Psi}_{1,j}(r, z, z_0)$ and the next obtained integration of the series obtained. The integration proceeds along the contour obtained by deforming the original integration path into the line of most rapid decrease $\exp[i\phi_{N,j}(r, z, z; \kappa)]$ with an increase of $|\kappa - \kappa_{N,j}|$. This line passes through

saddle point $\kappa_{N,j}$ and ends at the origin of function $p(\kappa)$. Function $\Psi_{N,j}(r, z, z_0)$ describes a wave corresponding to a geometric beam reflected N times from the boundary $z = H$.

The expression in the right side of (22a) is considerably less than unity, even with small values of $\tilde{R}_{1,k} - r)$. It is easy to be convinced that $|Q_{1,1}| \ll 1$ is also much less than 1 at $r = R_{1,k}$.

Having calculated the values of $\Psi_0(r, z, z_0)$ and $\tilde{\Psi}_1(r, z, z_0)$ by the method shown above we can now find the complete field $\Psi(r, z, z_0)$, from formula (5), in the geometric refraction shadow zone. In Fig. 3 there is shown the dependence of function $\Psi(r, z, z_0)$ on r, calculated for the particular case: $a = u \cdot 10^{-4}$ 1/m, $n - z = 100$ m, $H - z = 800$ m under the assumption that $\rho_g = \rho$; under this condition the contribution of the waves reflected from the horizon $z = H$ to field $\Psi(r, z, z_0)$ is negligibly small. In the graph of Fig. 3, the points mark the values of $\Psi(r, z, z_0)$, calculated by the normal-wave method. The results obtained by both methods give satisfactory agreement.

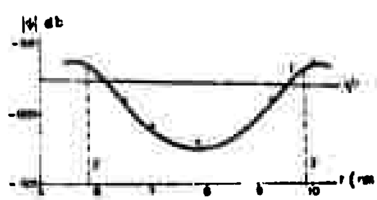

Fig. 3. Graph of the dependence of function $\Psi(r, z, z_0)$ on r.
1 - Values of $|\Psi|$ calculated by the normal-wave method; 2 - shadow-zone boundaries.

Let us analyze the case when $n(z) < n(H)$, $u(H, n(z)) > 1$. Here the integral of formula (6b), which describes $\tilde{\Psi}_1(r, z, z_0)$, may conveniently be estimated by the stationary-phase method.

Let us formally expand the integrand in formula (6b) into a series of powers of $(p(\kappa)q(\kappa))$. The terms of this series have real stationary-phase points only at $\kappa \in [0, n(z)]$, for which

$$\frac{F_1(r, z, z_0; \kappa)}{1 - p(\kappa)\, q(\kappa)} = \sum_{N=1}^{\infty} \sum_{j=2}^{5} F_{1,j}(r, z, z_0; \kappa)\, [p(\kappa)\, q(\kappa)]^{N-1}, \tag{26}$$

while $F_{1,j}(r, z, z_0; \kappa)$ satisfy equation (16b). The essential stationary-phase points $\kappa_{N,j}$ which correspond to a term of the series in the right side of formula (26) are roots of equation (25).

Let us plot the integration contour so that $\operatorname{Im} \kappa = \lim_{\varepsilon \geq 0} (+\varepsilon)$ on it. Then

$$\tilde{\Psi}_1(r, z, z_0) = \sum_{N=1}^{\infty} \sum_{j=2}^{5} \Psi_{N,j}(r, z, z_0) + \sum_{m=1}^{M} \hat{\Psi}_m(r, z, z_0). \tag{27}$$

Here $\Psi_{N,j}(r, z, z_0)$ corresponds to the contribution of stationary-phase points of a term of the series of formula (26).

If equation (25) has a unique root $\kappa_{N,j}$, then $\Psi_{N,j}(r, z, z_0)$ is described by equality (24).

But $R_{N,j}(z, z_0; \kappa)$, where $j = 3, 5$, as a function of real values of κ, has a maximum at point $\kappa = \kappa_{N,j,k}$, which corresponds to the caustic of the beams reflected from the horizon $z = H$. This caustic lies completely in the region of values of z for which $n(z) < n(H)$. Magnitude $\kappa_{N,j,k}$ is the root of equation $\frac{\partial}{\partial \kappa} R_{N,j}(z, z_0; \kappa) = 0$. The distance from the sound source to the caustic along the horizontal is equal to

$$R_{N,j,k} = R_{N,j}(z, z_0; \kappa_{N,j,k}).$$

If r is close to $R_{N,j,k}$ and $r > R_{N,j,k}$, then

$$\Psi_{N,j}(r, z, z_0) = \frac{k_0^{1/6} \cdot 2^{5/6} \kappa_{N,j,k}^{1/2}\, V(t)\, (1 + Q_{N,j}) \exp[i\varphi_{N,j}(r, z, z_0; \kappa_{N,j,k})]}{r^{1/2} \left[\frac{\partial^2}{\partial \kappa^2} R_{N,j}(z, z_0; \kappa)_{\kappa=\kappa_{N,j,k}}\right]^{1/3} \sqrt[4]{n^2(z_0) - \kappa_{N,j,k}^2}\, \sqrt[4]{n^2(z) - \kappa_{N,j,k}^2}}, \tag{26a}$$

where

$$t = 2^{1/3} k_0^{2/3} (r - R_{N,j,k}) \left| \left[\frac{\partial^2}{\partial \kappa^2} R_{N,j}(z, z_0; \kappa)_{\kappa=\kappa_{N,j,k}}\right]^{-1/3} \right|.$$

Function $\sum_{m=1}^{M} \grave{\Psi}_m(r, z, z_0)$ is the sum of normal waves corresponding to real poles $\kappa_m \in [n(h), n(H)]$. A normal wave is defined as the residue in pole κ_m of the integrand in formula (1).

With the given form of $n^2(z)$, $\hat{\Psi}_m(r, z, z_0)$ is described by the formulas given in work [2]. The function $\Sigma\hat{\Psi}_m(r, z, z_0)$ gives the basic contribution to $\tilde{\Psi}_1(r, z, z_0)$, if $|V| \ll 1$ or $|V| < 1$, $r \to \infty$, and characterizes the diffraction "exposure" of the geometric refraction shadow zone stretching limitlessly in the positive direction of the r axis on horizon z under consideration.

Although at $n(z) < n(H)$ horizon z does not pass through the secondary and subsequent illuminated zones, the resultant field $\Psi(r, z, z_0)$ will nevertheless be characterized on it by alternating maxima and minima. The picture is externally similar to that which takes place at $n(z) > n(H)$. But in the case in question the appearance of maximum $|\Psi(r, z, z_0)|$ is caused by the approximation to the caustic of the beams reflected from the ground, instead of the input to the illuminated zone. Such a false "illuminated zone" differs from the true one in that maximum $|\Psi(r, z, z_0)|$ in this zone depends on the frequency of the sound, even with the consideration of absorption in the medium.

The acoustic field in the channel is represented in the form of a finite number of components, each of which is described by a converging integral. This representation is especially convenient in the region where the field has a zonal structure, since it permits us to calculate the field here faster and simpler than by the normal-wave method ordinarily used for this purpose.

The field was calculated in the refraction geometric shadow zone in which $n^2(z) = 2 \exp[a(h - z)] - \exp[2a(h - z)]$. It is important to note that the obtained formulas are also valid with another similar dependence of $n^2(z)$, since the specific form of $n^2(z)$, as a rule, is not taken into account in the derivation of the formulas.

References

1. Brekhovskikh, L. M. Volny v sloistykh sredakh (Waves in laminar media), Izd-vo AN SSSR, Moscow, 1957.

2. Masterov, Ye. P. K voprosu o volnovodnom rasprostranenii zvuka v sloisto-neodnorodnykh sredakh (The problem of waveguide expansion of sound in laminar-heterogeneous media), Akust. zh., Vol. 5, No. 3, 1959.

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ABSTRACT

(U) A method for calculating acoustic fields in channels is presented. It is assumed that the channel is filled with a medium which has no shear elasticity and which is nonhomogeneous in the laminar sense. The method is based on the postulate that the selection of the field components, (the sum of which describes this field), closely correspond to the spatial structure of the acoustic field, thus facilitating the computation. The acoustic field in the channel is described by a finite number of components, each of which is in turn described by a converging integral. This technique is particularly useful when the acoustic field has a zonal structure, allowing for faster and simpler calculations, than it would be possible using the conventional method of plane waves. The acoustic field was computed for the region of the refractive geometric shadow in the channel. Orig. art. has: 2 figures, 26 formulas.